ELASTIC BENDING VIBRATIONS OF A ROD CARRYING ELECTRIC CURRENT

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ABSTRACT: In [1] a dispersion relation for the vibrations of an elastic rod of circular cross section with an electric current flowing over its surface was obtained, and a detailed study was made of the particular case of axially symmetric vibrations. The present paper is devoted to an examination of the longwave bending vibrations of elastic rods with an electric current flowing over their surface. These vibrations are of special interest since they have the lowest frequency and hence the last stability.

1. Investigation of the bending vibrations of a circular rod on the basis of the general equations of the theory of elasticity. We shall consider a perfectly conducting solid rod of radius a with free ends and a constant current I flowing over its surface. Let the displacements of points on the rod be described by the vector

$$\mathbf{u} = \mathbf{U}(r) \exp i(-\omega t + m\theta + kz).$$

The case of axially symmetric vibrations with m = 0 was thoroughly investigated in [1]. Here we shall consider the case of bending vibrations (m = 1). For an infinite rod the dispersion relation for such vibrations has the form

$$|d_{ij}| = 0 \qquad (i, j = 1, 2, 3), \tag{1.1}$$

with the following elements of the determinant:

$$d_{11} = 2h^2[1 + \psi_1(x)] + \frac{1}{1+\nu} + \left[x^2\left(y^2 - \frac{1}{1+\nu}\right) - \frac{1}{1+\nu}\right]\varphi_1(X),$$

$$d_{12} = \frac{Y^2 - 1}{1+\nu}\varphi_1(Y) - \left[x^2\left(y^2 - \frac{1}{1+\nu}\right) - \frac{1}{1+\nu}\right]\varphi_1(X),$$

$$d_{13} = -\frac{1}{(1+\nu)\varphi_1(Y)} - \left[x^2\left(y^2 - \frac{1}{1+\nu}\right) - \frac{1}{1+\nu}\right]\varphi_1(X),$$

$$d_{21} = 1 - \varphi_1(X), \qquad d_{22} = \varphi_1(X) - \varphi_1(Y),$$

$$d_{23} = \varphi_1(X) - \frac{Y^2}{2} - \frac{1}{\varphi_1(Y)}, \qquad d_{31} = h^2 - \frac{1}{1+\nu},$$

$$d_{32} = y^2, \qquad d_{33} = \frac{1}{2(1+\nu)}, \tag{1.2}$$

Here

$$X^2 = k^2a^2\left(\frac{\rho\omega^2}{k^2(\lambda + 2\mu)} - 1\right) = x^2\left(y^2\frac{(1+\nu)(1-2\nu)}{1-\nu} - 1\right)$$

$$Y^2 = k^2a^2\left(\frac{\rho\omega^2}{k^2\mu} - 1\right) = x^2(2y^2(1+\nu) - 1),$$

$$h^2 = \frac{H_0^2}{8\pi E} = \frac{I^2}{200\pi a^2 E}, \qquad H_0 = \frac{2I}{10a}$$

$$\varphi_1(\xi) = \frac{J_1(\xi)}{\xi J_1'(\xi)}, \qquad \psi_1(\xi) = \frac{K_1(\xi)}{\xi K_1'(\xi)},$$

$$x = ka, \qquad y^2 = \frac{\rho\omega^2}{Ek^2}$$

where E is the modulus of elasticity, λ, μ the Lamé coefficients, ν Poisson's ratio, ρ the density of the material, I the current in amps, H_0 the magnetic field at the surface of the conductor at $r = a$, and $J_1(\xi)$, $K_1(\xi)$ are cylindrical functions.

Solving (1.1) for h^2, after certain transformations we get

$$h^2 = \frac{x^2}{1+\nu}\frac{|a_{ij}|}{|b_{ij}|} \qquad (i, j = 1, 2, 3). \tag{1.3}$$

The elements of the determinant in (1.3) have the form

$$a_{11} = y^2(1+\nu) - 1,$$

$$a_{12} = a_{13} = 2y^2(1+\nu) - 1, \qquad a_{21} = \varphi(X) - 2,$$

$$a_{23} = -2[\varphi(Y) - 2] - x^2[2y^2(1+\nu) - 1],$$

$$a_{22} = \varphi(Y) - 2, \qquad a_{33} = 1,$$

$$a_{31} = \varphi(X) - 1, \qquad a_{32} = -[y^2(1+\nu) - 1][\varphi(Y) - 1],$$

$$b_{11} = 2[1 + \psi_1(x)],$$

$$b_{12} = (Y^2 - 1)[\varphi(X) - 1] -$$

$$\{x^2[y^2(1+\nu) - 1] - 1\}[\varphi(Y) - 1],$$

$$b_{13} = -2(Y^2 - 1)/[\varphi(Y) - 1] - 2[\varphi(Y) - 1],$$

$$b_{21} = 0, \qquad b_{22} = \varphi(Y) - \varphi(X),$$

$$b_{23} = 2/[\varphi(Y) - 1] - Y^2 - 2[\varphi(Y) - 1],$$

$$b_{33} = 1 - 2y^2(1+\nu),$$

$$b_{31} = 1, \qquad b_{32} = y^2(1+\nu)[\varphi(X) - 1][\varphi(Y) - 1],$$

$$\varphi(\xi) = \xi J_0(\xi)/J_1(\xi). \tag{1.4}$$

In the case of long waves ($ak \ll 1$) for the functions $\varphi(\xi)$ and $\psi_1(\xi)$ we have the following approximate expressions:

$$\varphi(\xi) \approx 2 - \xi^2/4, \qquad 2[1 + \psi_1(\xi)] =$$

$$= -2K_0(\xi)/K_1'(\xi) \approx 2\xi^2 \ln(2/\gamma\xi), \tag{1.5}$$

$$K_1(\xi) \approx 1/\xi, \qquad K_0(\xi) \approx -(\ln(\xi/2) + C).$$

Here $\ln\gamma = C \approx 0.577$ is Euler's constant. Using these expansions, from (1.3) we get the dispersion relation for longwave vibrations:

$$\frac{\omega^2\rho}{k^2E} = \frac{1}{4}a^2k^2 + \frac{I^2}{100\pi a^2E}\left(\ln\frac{ka}{2} + C + \frac{1}{2}\right). \tag{1.6}$$

2. Approximate theory of longwave bending vibrations of an elastic rod carrying current. a) General relations. We shall consider a homogeneous cylindrical rod of arbitrary but constant cross section and infinite length. We shall assume that an electric current I flows over the surface of this rod. If the wavelength of the bending vibrations is much greater than the rod diameter, and the vibrations themselves are plane, then the equation of the vibrations may be written in the form [2]

$$\rho A\frac{\partial^2 w}{\partial t^2} = -EJ\frac{\partial^4 w}{\partial z^4} + f_y + \ldots, \tag{2.1}$$

Here A is the cross-sectional area of the rod; the wave is propagated in the direction of the rod axis z; w is the displacement in the direction of the y axis, perpendicular to the axis of the rod, J is the moment of inertia of the rod cross section, and f_y the external force acting on unit length of the rod in the direction of the displacement w. In the case of longwave vibrations the discarded terms in (2.1) have a higher order of smallness. In the case of a current-carrying rod the force f_y owes its manifestation to the magnetic field. If we take two sections perpendicular to the axis of the undeformed rod and a distance dl_0 apart, then

$$\mathbf{f} = -\frac{1}{8\pi dl_0}\int_{(S)} H^2\mathbf{n}\,dS \approx -\frac{1}{8\pi}\oint_{(s)} H^2\mathbf{n}\left(1 + \frac{y}{R}\right)ds. \tag{2.2}$$

Here H is the intensity of the magnetic field at the surface of the deformed rod, **n** is the outward normal to the lateral surface of the isolated element of the deformed conductor, dS is an area element of the lateral surface, ds is a contour element of the cross section, and R is the radius of curvature of the rod axis; the integrals are taken over the lateral surface and over the closed contour of the cross section, respectively.

In the case of small deformations we may assume that

$$\frac{1}{R} = -\frac{\partial^2 w(z, t)}{\partial z^2} = -w'', \qquad \mathbf{H} = \mathbf{H}_0 + \mathbf{H}_1, \tag{2.3}$$

where H_1 is the perturbation of the magnetic field. Then expression (2.2) for the force **f** can be linearized with respect to perturbations of the magnetic field [3]. For f_y we have

$$f_y = \frac{w''}{8\pi}\oint_{(s)} H_0^2 n_y y\, ds - \frac{1}{4\pi}\oint_{(s)} \mathbf{H}_0\cdot\mathbf{H}_1 n_y\, ds \equiv f_1 + f_2\,. \qquad (2.4)$$

Here we have taken into account the fact that in the absence of vibrations $\mathbf{H} = \mathbf{H}_0$, $\mathbf{f} = 0$, while H_1 denotes the value of the magnetic field perturbation at the surface S. From (2.4) it is clear that the calculation of f_2 presupposes determination of the field perturbation H_1.

In computing the perturbed field **H** it is natural to use the scalar potential Φ: $\mathbf{H} = \nabla\Phi$, where Φ satisfies the Laplace equation $\Delta\Phi = 0$. On the surface of the conductor, in virtue of the assumption made above, that the entire current flows over the surface, the field must satisfy the boundary condition

$$\mathbf{H}\cdot\mathbf{n} = 0\,. \qquad (2.5)$$

Let the deformed axis of the rod be described by the equation $y = w(x, t)$. We shall go over to a new coordinate system X, Y, Z linked with the old one by the relations

$$X = x, \qquad Y = y - w, \qquad Z = z.$$

In the new system the equation $\Delta\Phi = 0$ takes the form

$$\Delta_{X,Y,Z}\Phi = \frac{\partial^2 w}{\partial Z^2}\frac{\partial\Phi}{\partial Y} + 2\frac{\partial w}{\partial Z}\frac{\partial^2\Phi}{\partial Y\partial Z} - \left(\frac{\partial w}{\partial Z}\right)^2\frac{\partial^2\Phi}{\partial Y^2}\,. \qquad (2.6)$$

We shall solve this equation by the method of perturbations. We set $\Phi = \varphi_0 + \varphi_1$. The potential of the unperturbed field satisfies the equation

$$\Delta_{X,Y}\varphi_0 = 0. \qquad (2.7)$$

Assuming that the bending of the conductor is small, we shall neglect terms of the second order of smallness in w, φ_1 and their derivatives.

Then for φ_1 from (2.6) we get the equation

$$\Delta_{X,Y,Z}\varphi_1 = \frac{\partial^2 w}{\partial Z^2}\frac{\partial\varphi_0}{\partial Y}\,. \qquad (2.8)$$

The solution φ_1 of Eq. (2.8) can be represented as the sum of the solutions φ of the homogeneous equation and the particular solution φ' of the inhomogeneous equation

$$\varphi_1 = \varphi + \varphi'. \qquad (2.9)$$

In connection with the decomposition (2.9) of the field perturbation into two parts, it is likewise natural to divide the force f_2 in (2.4) into two components,

$$f_2 = f' + f''. \qquad (2.10)$$

The inhomogeneous equation has the particular solution

$$\varphi' = w\frac{\partial\varphi_0}{\partial Y}\,. \qquad (2.11)$$

The solution of the homogeneous equation

$$\left(\Delta_{X,Y} + \frac{\partial^2}{\partial Z^2}\right)\varphi = 0 \qquad (2.12)$$

is uniquely determined by boundary condition (2.5). Setting

$$\varphi = w\Psi(X, Y), \qquad w = w_0 \exp ikZ \qquad (w_0 = w_0(t))\,, \qquad (2.13)$$

we can rewrite (2.12) in the form

$$(\Delta_{X,Y} - k^2)\,\Psi = 0\,. \qquad (2.14)$$

Since in deriving (2.1) it was assumed that $k \to 0$, for our purposes Eq. (2.14) can be solved approximately, with account only for the terms containing lower powers of k^2. However, in simple cases it is more convenient to start from exact solutions of (2.14) and perform the expansion in powers of k^2 in the final formulas. This is the method used below.

b) Bending vibrations of a circular rod. Using the cylindrical system of coordinates R, ϑ, Z($X = R\cos\vartheta$, $Y = R\sin\vartheta$), we can write the scalar potential of the unperturbed magnetic field of a circular rod φ_0, the solution of Eq. (2.7), as

$$\varphi_0 = {}^2/_{10}\, I\vartheta \qquad (H_0 = 2I\,/\,10a). \qquad (2.15)$$

Here I is the current flowing over the surface of the conductor in amps, H_0 is the intensity of the unperturbed magnetic field at the surface of the conductor, and a is the radius of the conductor. With the help of (2.15) we will find the particular solution of (2.11) for the perturbation of the magnetic field:

$$\varphi' = w\frac{\partial\varphi_0}{\partial Y} = w\frac{2I}{10R}\cos\vartheta\,. \qquad (2.16)$$

The corresponding solution of the homogeneous equation (2.14) will be

$$\varphi = BwK_1(kR)\cos\vartheta \qquad (B = 2I\,/\,10akaK_1'(ka))\,. \qquad (2.17)$$

The constant B was determined from condition (2.5), the prime denotes the derivative with respect to the total argument. Substituting expressions (2.15), (2.16), and (2.17) and (2.4) and (2.10), and also bearing in mind that $n_Y = \sin\vartheta$, $Y = a\sin\vartheta$, $ds = a\,d\vartheta$, we get

$$f_1 = {}^1/_{200}I^2w'', \qquad f' = {}^1/_{100}I^2w\,/\,a^2,$$
$$f'' = {}^1/_{100}I^2wK_1(ka)\,/\,a^2kaK_1'(ka)\,. \qquad (2.18)$$

Since $f_y \equiv f_1 + f_2 = f_1 + f' + f''$, adding the components of the force (2.18), using (1.5), and assuming that

$$w \sim \exp i(-\omega t + kZ)\,, \qquad (2.19)$$

we find the force acting on unit length of the rod,

$$f_y = -{}^1/_{100}I^2wk^2(\ln {}^1/_2\, ka + C + {}^1/_2) + O(k^2)\,. \qquad (2.20)$$

Substituting (2.20) into (2.1), we get the dispersion relation for the bending vibrations of a circular rod

$$\rho A\omega^2 = EJk^4 + {}^1/_{100}I^2k^2(\ln {}^1/_2\, ka + C + {}^1/_2)\,. \qquad (2.21)$$

If we consider that the moment of inertia of a circle $J = 1/4\pi a^4$, then Eq. (2.21) coincides with Eq. (1.6).

c) Bending vibrations of a thin conductor of elliptical cross section. 1. Let the contour equation of the conductor cross section be described in the coordinate system X, Y by the equation

$$X^2\,/\,a^2 + Y^2\,/\,b^2 = 1\,. \qquad (2.22)$$

We introduce the elliptical coordinates ξ, η with the help of the relation

$$\zeta = X + iY = h\,\mathrm{ch}\,(\xi + i\eta)$$
$$\begin{pmatrix} h^2 = a^2 - b^2, & a = h\,\mathrm{ch}\,\xi_0, & b = h\,\mathrm{sh}\,\xi_0 \\ 0 \leqslant \xi < \infty, & -\pi \leqslant \eta < \pi, & \xi_0 = \ln[(a+b)/h] \end{pmatrix}. \qquad (2.23)$$

The function that realizes the conformal mapping of the exterior of the ellipse (2.22) onto the exterior of the unit circle has the form

$$W = \frac{\zeta + \sqrt{\zeta^2 - h^2}}{a + b}\,. \qquad (2.24)$$

The magnetic field of the unperturbed cylinder can be described both by the scalar potential φ_0 and by the z-component of the vector potential A_Z,

$$\mathbf{H}_0 = \nabla\varphi_0, \qquad \mathbf{H}_0 = \mathrm{rot}\,(\mathbf{z}_0 A_z)\,. \qquad (2.25)$$

The complex potential F of the magnetic field of a straight conductor is

$$F = A_z + i\varphi_0 = 0.2\, I \ln W\,. \qquad (2.26)$$

Substituting expressions (2.23) and (2.24) into (2.26), we get

$$\varphi_0 = 0.2\, I\eta\,. \qquad (2.27)$$

Hence we find the strength of the magnetic field at the surface of the unperturbed elliptical conductor

$$H_0 = \frac{2I}{10h\,(\operatorname{ch}^2\xi_0 - \cos^2\eta)} \tag{2.28}$$

and the particular solution (2.11)

$$\varphi' = \frac{2Iw}{10h}\,\frac{\operatorname{sh}\xi\cos\eta}{\operatorname{ch}^2\xi - \cos^2\eta}\,. \tag{2.29}$$

2. In elliptic coordinates Eq. (2.14) is written in the form

$$\left[\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - k^2h^2(\operatorname{ch}^2\xi - \cos^2\eta)\right]\Psi = 0\,. \tag{2.30}$$

We will represent the solutions of (2.30) as $\Psi = \chi(\xi)\psi(\eta)$. Then [4,5]

$$\chi''(\xi) + (a + 16q\operatorname{ch}2\xi)\,\chi(\xi) = 0 \tag{2.31}$$

$$\psi''(\eta) - (a + 16q\cos 2\eta)\,\psi(\eta) = 0 \qquad \left(q \equiv \frac{k^2h^2}{32}\right). \tag{2.32}$$

From the requirement of periodicity of $\psi(\eta)$ we find the discrete eigenvalues $a_p(q)$ and the corresponding Mathieu functions. In our case these will be even functions of odd index:

$$\psi_p = ce_{2n+1}(\eta, q), \quad p \equiv 2n+1\,. \tag{2.33}$$

As $\chi_p(\xi)$ we must take the solution that satisfies the condition

$$\chi_p(\xi) \to 0 \quad \text{as} \quad \xi \to \infty\,.$$

This property is possessed by the so-called $\mathrm{Fek}_{2n+1}(\xi, -q)$ functions, which we shall denote by $Q_{2n+1}(\xi, q)$. The general solution of Eq. (2.30) may be written thus:

$$\varphi = w\Psi = w\sum_{p\equiv 2n+1=1}^{\infty} b_p Q_p(\xi, q)\, ce_p(\eta, q)\,. \tag{2.34}$$

In connection with the fact that in the sense of approximation (2.1) the quantity

$$q \equiv {}^1/_{32}k^2h^2 \ll 1\,, \tag{2.35}$$

in the Fourier expansions of the Mathieu functions (2.33) it is sufficient to confine oneself to terms of the expansion containing q in the zeroth and first powers [4]. Therefore, we may assume that

$$ce_{2n+1}(\eta, q) \approx A_{2n+1}^{(2n+1)}\cos(2n+1)\eta +$$
$$+ A_{2n+3}^{(2n+1)}\cos(2n+3)\eta + A_{2n-1}^{(2n+1)}\cos(2n-1)\eta$$
$$(A_p^{(p)} \approx 1, \quad A_{p\pm 2}^{(p)} \approx q)\,. \tag{2.36}$$

3. The coefficients b_p in (2.34) are determined from the boundary condition $H \cdot n = 0$, i.e.,

$$\frac{\partial\varphi_1}{\partial\xi} = \frac{\partial}{\partial\xi}(\varphi + \varphi') = 0 \text{ when } \xi = \xi_0 = \ln[(a+b)/h]\,. \tag{2.37}$$

Taking into account (2.29) and setting

$$\frac{\partial}{\partial\xi}\,\frac{\operatorname{sh}\xi\cos\eta}{\operatorname{ch}^2\xi - \cos^2\eta} \equiv -\sum_{p=1}^{\infty}\beta_p ce_p(\eta, q) \tag{2.38}$$

for $\xi = \xi_0$, we find

$$\varphi_1 = \frac{2I}{10h}\,w\sum_{p\equiv 2n+1=1}^{\infty}\beta_p\,\frac{Q_p(\xi, q)}{(\partial Q_p/\partial\xi)_{\xi=\xi_0}}\,ce_p(\eta, q)\,. \tag{2.39}$$

We note that if $ce_p(\eta, q)$ can be written in the form (2.36), then the coefficients β_p in (2.38), (2.39) are expressed in terms of the coefficients of the Fourier expansion α_p

$$\frac{\partial}{\partial\xi}\,\frac{\operatorname{sh}\xi\cos\eta}{\operatorname{ch}^2\xi - \cos^2\eta} = -\sum_{p=1}^{\infty}\alpha_p\cos pn \qquad (\xi = \xi_0) \tag{2.40}$$

by the relations

$$\beta_p = \alpha_p + \alpha_{p-2}A_{p+2}^{(p-2)} + \alpha_{p+2}A_{p-2}^{(p+2)}\,. \tag{2.41}$$

The values of α_p in series (2.40) are found in elementary fashion, but the expressions obtained are clumsy and are not given here.

4. Knowing the expressions for the fields, we now turn to computing the forces f_1 and $f_2 = f' + f''$. These computations may also conveniently be performed in elliptical coordinates. In this case

$$ds = h\sqrt{\operatorname{ch}^2\xi_0 - \cos^2\eta}\,d\eta, \qquad n_Y = \frac{\operatorname{ch}\xi_0\sin\eta}{\sqrt{\operatorname{ch}^2\xi_0 - \cos^2\eta}},$$
$$Y = h\operatorname{sh}\xi_0\sin\eta, \qquad H_1 = \frac{1}{h\sqrt{\operatorname{ch}^2\xi_0 - \cos^2\eta}}\,\frac{\partial\varphi_1}{\partial\eta}\,. \tag{2.42}$$

Substituting (2.42) into the formula for f_1 we obtain, using (2.23) and (2.28), the expression

$$f_1 = {}^1/_{100}I^2w''b\,/\,(a+b)\,. \tag{2.43}$$

Integrating by parts, we reduce Eq. (2.4) for $f_2 = f' + f''$ to the form

$$f_2 = \frac{I}{2\pi 10h}\int_0^{2\pi}\varphi_1\frac{\partial}{\partial\eta}\,\frac{\operatorname{ch}\xi_0\sin\eta}{\operatorname{ch}^2\xi_0 - \cos^2\eta}\,d\eta\,. \tag{2.44}$$

The computation of f_2 is appreciably simplified by the fact that

$$\frac{\partial}{\partial\eta}\,\frac{\operatorname{ch}\xi\sin\eta}{\operatorname{ch}^2\xi - \cos^2\eta} \equiv -\frac{\partial}{\partial\xi}\,\frac{\operatorname{sh}\xi\cos\eta}{\operatorname{ch}^2\xi - \cos^2\eta}\,. \tag{2.45}$$

Consequently, after very simple calculations, using (2.39), we find

$$f' = {}^1/_{200}I^2w\,(1/a^2 + 1/b^2)\,, \tag{2.46}$$

$$f'' = \frac{I^2}{100}\,\frac{w}{h^2}\sum_{p=1}^{\infty}\beta_p^2(q)\,\frac{Q_p(\xi_0, q)}{(\partial Q_p/\partial\xi)_{\xi=\xi_0}}\,. \tag{2.47}$$

5. From (2.1) and (2.43), (2.46), and (2.47) with condition (2.19) we get the dispersion relation

$$\rho A\omega^2 = EJk^4 + \frac{I^2}{100}\left[\frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) -\right.$$
$$\left. - k^2\frac{b}{a+b} + \frac{1}{h^2}\sum_{p=1}^{\infty}\beta_p^2(\xi_0, q)\,\frac{Q_p(\xi_0, q)}{(\partial Q_p/\partial\xi)_{\xi=\xi_0}}\right]. \tag{2.48}$$

6. When the eccentricity is small, i.e., $h/a \ll 1$, expression (2.47) for f'' can be evaluated explicitly. In fact, expanding the left side of (2.38) in a Fourier series and neglecting terms of higher order of smallness than

$$e^{-2\xi} \approx h^2/a^2 \tag{2.49}$$

we get

$$\frac{\partial}{\partial\xi}\,\frac{\operatorname{sh}\xi\cos\eta}{\operatorname{ch}^2\xi - \cos^2\eta} \approx 2e^{-\xi_0}\{\cos\eta + 3e^{-2\xi_0}\cos 3\eta\}. \tag{2.50}$$

The expansions of the functions ce_p have the form

$$ce_1(\eta) = \cos\eta + q\cos 3\eta + \ldots, \quad ce_3(\eta) = \cos 3\eta + \ldots \tag{2.51}$$

and, consequently,

$$\beta_1 = -\frac{2h}{a+b}\,, \qquad \beta_3 = -\frac{2h}{a+b}\left(-q + 3\,\frac{h^2}{(a+b)^2}\right). \tag{2.52}$$

Here we have taken into consideration the fact that $e^{\xi_0} = (a+b)/h$. It is now clear that, with the accuracy assumed, in (2.47) there remains only the first term

$$f'' \approx \frac{I^2}{100}\,w\,\frac{4}{(a+b)^2}\,\frac{Q_1}{(\partial Q_1/\partial\xi)_{\xi=\xi_0}}\,. \tag{2.53}$$

Using the approximation

$$\frac{4}{(a+b)^2} \approx \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \tag{2.54}$$

we obtain (see (2.46) and (2.53))

$$f_2 = f' + f'' = 4\,\frac{wI^2}{100\,(a+b)^2}\left(1 + \frac{Q_1}{(\partial Q_1/\partial\xi)_{\xi=\xi_0}}\right). \qquad (2.55)$$

7. The expression for Q_1 can be found as follows: substituting in (2.31) for $\chi(\xi)$ the value $a_1 \approx 1 - 8q$ and considering that, with the assumed accuracy,

$$\mathrm{ch}\, 2\xi \approx {}^1/_2 e^{2\xi}, \qquad (2.56)$$

we find the approximate equation for $Q_1(\xi)$

$$Q_1''(\xi) - (1 - 8q + 8qe^{2\xi})\, Q_1(\xi) = 0. \qquad (2.57)$$

This Bessel equation with fractional index has the following solution with the required behavior as $\xi \to \infty$:

$$Q_1(\xi) \approx K_{\sqrt{1-8q}}\,(\sqrt{8q}\,e^{\xi}) \approx K_{1-4q}\,(\sqrt{8q}\,e^{\xi}). \qquad (2.58)$$

Starting from the definition

$$K_n = \frac{\pi}{2\sin\pi n}\,[I_{-n} - I_n], \quad I_n = \sum_{r=0}^{\infty}\left(\frac{z}{2}\right)^{n+2r}\frac{1}{r!\,\Gamma(n+r+1)}, \qquad (2.59)$$

we can obtain

$$Q_1(\xi) = K_{1-4q}(\xi) \approx K_1(\xi) + 4q\left[\frac{1}{\xi}\ln\frac{\xi}{2} + \frac{C}{2}\right]. \qquad (2.60)$$

Hence, with (2.23) and (2.35), we find

$$1 + \frac{Q_1(\xi)}{(\partial Q_1/\partial\xi)_{\xi=\xi_0}} = -\frac{k^2(a+b)}{4}\,2b\left[\ln\frac{k(a+b)}{4} + \frac{a+b}{2b}\,C + \frac{a-b}{2b}\right] \qquad (2.61)$$

and, consequently (see (2.55)),

$$f_1 = -\frac{wk^2}{100}\,I^2\,\frac{b}{a+b},\ f_2 = -w\,\frac{I^2}{100}\,\frac{2b}{a+b}\,k^2\left[\ln\frac{k(a+b)}{4} + C\,\frac{a+b}{2b} + \frac{a-b}{2b}\right]. \qquad (2.62)$$

Thus, the final expression for the total force will be

$$f_y = -w\,\frac{I^2}{100}\,\frac{2b}{a+b}\,k^2\left[\ln\frac{k(a+b)}{4} + C\,\frac{a+b}{2b} + \frac{a}{2b}\right]. \qquad (2.63)$$

When $a = b$ this formula goes over into (2.20). On making the substitution (2.63), we obtain dispersion relation (2.48) in the form

$$\rho A\omega^2 = EJk^4 + \frac{I^2}{100}\,k^2\,\frac{2b}{a+b}\left[\ln\frac{k(a+b)}{4} + C\,\frac{a+b}{2b} + \frac{a}{2b}\right]. \qquad (2.64)$$

3. Stability conditions for current-carrying rods. At certain values of the current I the frequency ω^2, linked with the current and the wavelength by Eq. (2.21) for circular rods and by Eq. (2.64) for elliptical rods, becomes negative. This means that there is a solution for ω that leads to an exponential growth of the random perturbations. Hence it follows that, starting from these current values, a straight conductor will be unstable.

a) For a solid circular conductor from dispersion relation (1.5) or from (2.21), after substituting the moment of inertia of a circle $J = \pi a^4/4$, we get the following expression for the value of the current at which the conductor loses stability:

$$I_* > \frac{10a^2k\,\sqrt{\pi E}}{2\sqrt{\ln(2/ka) - C - {}^1/_2}}. \qquad (3.1)$$

b) For an elliptical conductor from dispersion relation (2.64) we get the following expression for the value of the current at which the conductor loses stability:

$$I_* > \frac{10k\,\sqrt{a+b}\,\sqrt{EJ}}{\sqrt{2b}\,\sqrt{\ln[4/k(a+b)] - {}^1/_2\,a/b - {}^1/_2 C(a+b)/b}}. \qquad (3.2)$$

For vibrations of a solid conductor of elliptical cross section about the major axis of the ellipse $J = \pi ab^3/4$, for vibrations about the minor axis $J = \pi a^3 b/4$.

Example. Consider a circular conductor of radius $a = 0.1$ cm. Let a bending perturbation with wavelength L = 10 cm be propagated along it. In another case let $a = 1$ cm and L = 100 cm. Let $E = 10^6$ kg/cm^2. Then, for the first case from (3.1) we obtain $I_* > 39$ kA, for the second case $I_* > 386$ kA.

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